

# SMALL ORDERS OF HADAMARD MATRICES AND BASE SEQUENCES

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**ABSTRACT.** We update the list of odd integers  $n < 10000$  for which an Hadamard matrix of order  $4n$  is known to exist. We also exhibit the first example of base sequences  $BS(40, 39)$ . Consequently, there exist T-sequences  $TS(n)$  of length  $n = 79$ . The first undecided case has the length  $n = 97$ .

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## 1. INTRODUCTION

We ask a very simple question: For which odd positive integers  $n < 10000$  is it known how to construct an Hadamard matrix of order  $4n$ ? We shall refer to such  $n$  (in this range) as *good* integers, and to other as “bad”. Unfortunately, in spite of the fact that the Hadamard matrix conjecture is very old and constitutes a very active area of current research in combinatorics, the answer to this question is apparently not known. As a tentative answer we choose [1, Table 1.53]. In fact this table is more ambitious as it also provides the least exponent  $t$  for which it is known that an Hadamard matrix of order  $2^t n$  exists. We have qualified this answer as tentative for two reasons. First of all the table has been published three years ago and needs to be updated. For instance we have constructed Hadamard matrices of order 764 (see [6]). Secondly, the information contained in the table was not accurate even at the time of its publication. Indeed, in our note [7], we have given a list of 138 good values of  $n$ , which have not been recorded in the table. In the same note, by using new results, we have shown that 4 additional values of  $n$  are good.

We can now replace the above question with two simpler questions. First, is it true that the integers  $n$  asserted to be good in [1, Table 1.53] are indeed good? They probably are (and we continue to consider them as good) but I admit that I was not able to verify this assertion in all cases since the references provided are not sufficient and the literature

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on this subject is enormous. The tables in the old survey paper [15] are much better in that regard as they include the information necessary for the construction of tabulated matrices. The second simplified question is: can we convert some of the bad integers into good ones? We shall address only the latter question in this note.

We can state our main result simply by saying that we have converted 42 bad integers into good ones. (Most of them were good even three years ago.) Originally, i.e., according to [1, Table 1.53] there were 1006 bad integers. The update in [7] reduced this number to 864, and here we reduce it further to 822.

We refer the reader to [1, 15] and to our note [7] for the standard definitions and notation. As in that note, we shall write  $OD(4d)$  for the orthogonal design  $OD(4d; d, d, d, d)$ . It is now known that T-sequences of length  $d$  exist, and consequently  $OD(4d)$  exist, for all  $d \leq 100$  except possibly for  $d = 97$ . For this see the next section where we recall some old results and present a new one. We use these results later to construct some particular Hadamard matrices that we need.

## 2. TOOLS FOR THE CONSTRUCTION

Our objective is to show how one can construct Hadamard matrices of order  $4n$  for the following 42 odd integers  $n$ :

787, 823, 883, 1063, 1303, 1527, 2143, 2335, 2545, 2571,  
 3533, 4285, 5441, 5449, 5999, 6181, 6617, 6819, 7167, 7179,  
 7251, 7323, 7663, 7779, 8067, 8079, 8139, 8187, 8237, 8259,  
 8499, 8573, 8611, 8653, 8751, 8859, 9111, 9123, 9427, 9627,  
 9671, 9939.

(According to [1, Table 1.53] they are all bad.) This list does not overlap with the list of 142 good numbers in [7].

The construction is based on the following old results and on a new result that we will mention afterwards.

First, we need Mathon's theorem about symmetric conference matrices. Recall that a square matrix  $C$  of order  $n$  is called a *conference matrix* if its diagonal entries are 0, its off-diagonal entries are  $\pm 1$ , and  $CC^T = (n-1)I_n$ , where  $T$  denotes transposition and  $I_n$  is the identity matrix. If a conference matrix is symmetric, its order  $n$  must be 1 or  $\equiv 2 \pmod{4}$ .

**Theorem 2.1.** ( Mathon [12] ) If  $q \equiv 3 \pmod{4}$  is a prime power and  $q+2$  is a prime power, then there exists a symmetric conference

matrix of order  $q^2(q+2)+1$  and a symmetric Hadamard matrix of order  $2q^2(q+2)+2$ .

Next, we need three theorems of Yamada which we compress into two.

**Theorem 2.2.** ( Yamada [17] ) Let  $q \equiv 1 \pmod{8}$  be a prime power.

(a) If there exists an Hadamard matrix of order  $(q-1)/2$ , then there exists an Hadamard matrix of order  $4q$ .

(b) If there exists a symmetric conference matrix of order  $(q+3)/2$ , then there exists an Hadamard matrix of order  $4(q+2)$ .

Let us also recall that a *skew Hadamard matrix* is a Hadamard matrix  $H$  (of order  $n$  say) such that  $H - I_n$  is a skew-symmetric matrix.

**Theorem 2.3.** ( Yamada [17] ) If  $q \equiv 5 \pmod{8}$  is a prime power and there exists a skew Hadamard matrix of order  $(q+3)/2$ , then there exists an Hadamard matrix of order  $4(q+2)$ .

(For this theorem and part (b) of the previous one, Yamada gives credit to Z. Kiyasu.)

We also need two results of Miyamoto. The first one is the following theorem.

**Theorem 2.4.** ( Miyamoto [13] ) If  $q \equiv 1 \pmod{4}$  is a prime power and there exists an Hadamard matrix of order  $q-1$ , then there exists an Hadamard matrix of order  $4q$ .

For the second we need to recall the definition of Williamson-type matrices. Two matrices  $A, B$  of order  $n$  are *amicable* if  $AB^T = BA^T$ . Four  $\{+1, -1\}$ -matrices  $A, B, C, D$  of order  $n$  are called *Williamson-type matrices* if they are pairwise amicable and satisfy

$$AA^T + BB^T + CC^T + DD^T = 4nI_n.$$

We quote the second result of Miyamoto from the presentation provided by Seberry and Yamada [15, Corollary 8.8, part 1] or [14, Corollary 29, part (i)] where the proof is also given.

**Theorem 2.5.** *Let  $q \equiv 1 \pmod{4}$  be a prime power. Then there exist Williamson-type matrices of order  $q$  if there are Williamson-type matrices of order  $(q-1)/4$  or an Hadamard matrix of order  $(q-1)/2$ .*

The new result that we need is obtained by means of a computer. Namely, we have constructed the first example of base sequences  $BS(40, 39)$ . A well-known construction then gives us T-sequences of length 79, and also the orthogonal design  $OD(4d)$  with  $d = 79$ . This result will be used in the proof of Proposition 3.4 in the next section. We recall that

we have constructed in [9] T-sequences of length 73. Hence (see e.g., [1, Remark 8.47]) T-sequences  $TS(n)$  of length  $n \leq 100$  all exist except possibly for  $n = 97$ .

The base sequences  $(A; B; C; D) \in BS(40, 39)$  that we have found are given in encoded form by

$$[06582253432245718723, 11644426384657223422].$$

The encoding scheme is explained in [8]. For the reader's convenience we also give these sequences explicitly by writing  $+$  for  $+1$  and  $-$  for  $-1$ :

$$\begin{aligned} A &= ++--++--+-++++--+-+-- \\ &\quad ++--+-+++-++++-+++-+; \\ B &= +++--+------+++-+-- \\ &\quad +-+-+++-+-----; \\ C &= ++++++--++--++-++- \\ &\quad +-++++-+-----+; \\ D &= +++-----+++-+-----+ \\ &\quad -++--+-+--+---++-+++. \end{aligned}$$

The Base Sequence Conjecture (BSC) asserts that all  $BS(n+1, n)$  exist, i.e., are nonvoid. Due to the above example, we can now update its status (see [8]): BSC has been verified for all  $n \leq 40$  and is also known to be valid for all Golay numbers  $n = 2^a 10^b 26^c$  ( $a, b, c$  nonnegative integers).

### 3. EXISTENCE OF SOME HADAMARD MATRICES

For convenience, we split the proof into four propositions. We consider first the prime integers  $n$ .

**Proposition 3.1.** *For each of the primes*

$$n = 787, 823, 883, 1063, 1303, 2143, 3533, 5441, 5449, 8237, 8573$$

*there exists an Hadamard matrix of order  $4n$ .*

*Proof.* The case  $n = 787$  is an instance of Mathon's theorem. Indeed for  $q = 11$  we have  $q^2(q+2)+1 = 1574 = 2 \cdot 787$ .

In the cases  $n = 5441, 5449$  we apply part (a) of the first Yamada theorem with  $q = n$ . The existence of required Hadamard matrices of order  $(q-1)/2$  has been known for long time (see e.g. [15]).

In the case  $n = 883$  we use part (b) of the first Yamada theorem with  $q = n - 2 = 881$  (a prime). Then  $(q + 3)/2 = 442$  and, by Mathon's theorem, there exists a symmetric conference matrix of order 442.

In the cases  $n = 823, 1063, 1303, 2143$  we apply the second Yamada theorem with  $q = n - 2$ . The required skew Hadamard matrices of order  $(q + 3)/2 = 4 \cdot 103, 4 \cdot 133, 4 \cdot 163, 2^4 \cdot 67$  exist (see [2, 3, 5]).

In the remaining three cases  $n = 3533, 8237, 8573$  these primes are  $\equiv 5 \pmod{8}$  and we can apply the first Miyamoto theorem with  $q = n$ . This theorem requires  $q - 1 = 4d$  to be the order of an Hadamard matrix. If  $q = 3533, 8573$  we have  $d = 883, 2143$ . Both of these cases have been already handled in the previous paragraphs. In the remaining case 8237 we have  $d = 2059$ . We can easily handle this case since  $2059 = 29 \cdot 71$  and we know that there exist Williamson matrices of order 29 as well as T-sequences of length 71 (e.g., see [15] and [11] or [8]). This implies the existence of an Hadamard matrix of order  $4d$  (see [7]).  $\square$

(The first case,  $n = 787$ , could have been identified as good not only in [1, Table 1.53] but also in [15].)

**Proposition 3.2.** *For each of the numbers*

$$\begin{aligned} n = & 2571, 6819, 7179, 7251, 7323, 7779, 8067, 8139, \\ & 8187, 8259, 8499, 8859, 9087, 9123, 9939 \end{aligned}$$

*there exists an Hadamard matrix of order  $4n$ .*

*Proof.* Note that in all cases we have  $n = 3q$  where

$$\begin{aligned} q = & 857, 2273, 2393, 2417, 2441, 2593, 2689, 2713, 2729, 2753, \\ & 2833, 2953, 3041, 3209, 3313 \end{aligned}$$

is a prime  $\equiv 1 \pmod{4}$ . Since the orthogonal design  $OD(4d)$  exists for  $d = 3$ , it suffices to show that, for each of the 15 values of  $q$ , there exist Williamson-type matrices of order  $q$ . This can be deduced from the second Miyamoto theorem. Indeed, it suffices to verify that there exists an Hadamard matrix of order  $(q - 1)/2 = 4m$  where

$$m = 107, 284, 299, 302, 305, 324, 336, 339, 341, 344, 354, 369, 380, 401, 414.$$

For  $m = 107$  see [10] and for all other see [15].  $\square$

It is clear from this proof that all of these cases but the first could have been recorded in [1, Table 1.53].

**Proposition 3.3.** *For each  $n = 1527, 7167, 8079, 8751, 9111$  there exists an Hadamard matrix of order  $4n$ .*

*Proof.* We have  $n = 3q$  where  $q = 509, 2389, 2693, 2917, 3037$ . Again we use the  $OD(4d)$  for  $d = 3$  and our task is to show that there exist Williamson-type matrices of order  $q$ . As each of these  $q$  is a prime  $\equiv 1 \pmod{4}$ , we can use again the second Miyamoto theorem. This time we verify that there exist Williamson-type matrices of order  $(q-1)/4 = 127, 597, 673, 729, 759$ . For 127 see [4] and for all other see [15].  $\square$

**Proposition 3.4.** *For each of the numbers*

$$n = 2335, 2545, 4285, 5999, 6181, 6617, 7663, 8611, 8653, 9427, 9671$$

*there exists an Hadamard matrix of order  $4n$ .*

*Proof.* For  $n = 2335$  we shall apply the second Yamada theorem with  $q = 2333$ , a prime  $\equiv 5 \pmod{8}$ . We have to verify that there exists a skew Hadamard matrix of order  $(q+3)/2 = 2^4 \cdot 73$ . This is indeed true because there is an infinite series of skew Hadamard matrices constructed by E. Spence [16] which contains such a matrix of order  $4 \cdot 73$ .

Each of the remaining numbers is a product of two distinct primes, say  $n = dm$  with  $d < m$  and  $m = 97, 109, 509, 857, 883$ . (There are only 5 different  $m$ .) Since in all cases  $d < 97$ , we know that T-sequences of length  $d$  exist, and consequently also the orthogonal design  $OD(4d)$  exists. It remains to show that there exist Williamson-type matrices of order  $m$ . For  $m = 97$  and  $m = 109$  see [15] and for  $m = 509, 857, 883$  see the proofs of Propositions 3.3, 3.2 and 3.1, respectively.  $\square$

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